

GOOD COVERINGS OF ALEXANDROV SPACES

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ABSTRACT. In the present paper, we define a notion of good coverings of Alexandrov spaces with curvature bounded below, and prove that every Alexandrov space admits such a good covering and that it has the same homotopy type as the nerve of any good covering. We also prove the stability of the isomorphism classes of the nerves of good coverings in the non-collapsing case. In the proof, we need a version of Perelman’s fibration theorem, which is also proved in this paper.

1. INTRODUCTION

It is well known that there are relations between coverings and topology of spaces. In Riemannian geometry, Weinstein [25] found homotopy type finiteness of even-dimensional closed Riemannian manifolds of positively pinched curvature by covering those manifolds via convex balls whose number is uniformly bounded. Then Cheeger [4] extended this result to diffeomorphism finiteness by using a gluing method to a wider class of closed Riemannian manifolds with bounded sectional curvature. In the context of a lower sectional curvature bound, Grove and Petersen [6] succeeded to have a uniform bound on the number of metric balls, which are contractible *in* a larger concentric balls, needed to cover those Riemannian manifolds. See also [26], [8], [20] for related results. Covering methods are also useful to obtain bounds on the total Betti numbers. See [5], [27] for instance.

A covering of a topological space (resp. smooth manifolds) is called *good* if every nonempty finite intersection of elements in the covering is contractible (resp. diffeomorphic to an Euclidean space). See for instance [1]. In the present paper, we introduce a notion of good coverings of Alexandrov spaces with curvature bounded below.

Let M be an Alexandrov space with curvature bounded below. An open set U of M is called *conical* and strongly Lipschitz contractible (SLC in short) if it is homeomorphic to the tangent cone at a point $p \in U$ and is strongly Lipschitz contractible to p (see Sections 3 and 4 for precise definitions). An open set U is called convex if every minimal geodesic segment joining any two points of U is contained in U .

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We say that a locally finite covering $\mathcal{U} = \{U_i\}$ of M is *good* if every nonempty intersection $U_{i_0} \cap \cdots \cap U_{i_m}$ is a convex, conical SLC bounded domain.

The main results of the present paper are stated as follows.

Theorem 1.1. *For every open covering \mathcal{V} of an Alexandrov space M ,*

- (1) *there exists a locally finite refinement \mathcal{U} of \mathcal{V} which is a good covering.*
- (2) *M has the same homotopy type as the nerve of any good covering of it.*

Theorem 1.1(2) explicitly provides the homotopy type of any Alexandrov space from the information of a good covering.

Let $\mathcal{A}(n, D, v_0)$ denote the set of all isometry classes of n -dimensional compact Alexandrov spaces M with curvature ≥ -1 , $\text{diam}(M) \leq D$, $\text{vol}(M) \geq v_0 > 0$. We have the following stability of the nerves of good coverings of Alexandrov spaces.

Theorem 1.2. *There exist a positive number $\epsilon_0 = \epsilon_0(n, D, v_0)$, finitely many $M_1, \dots, M_N \in \mathcal{A}(n, D, v_0)$ and finite simplicial complexes K_1, \dots, K_N such that*

- (1) $\mathcal{A}(n, D, v_0) = \bigcup_{i=1}^N U_{GH}(M_i, \epsilon_0);$
- (2) *for any $M \in U_{GH}(M_i, \epsilon_0)$, there exists a good covering \mathcal{U}_M of M whose nerve is isomorphic to K_i .*

Here $U_{GH}(M, \epsilon)$ denotes the ϵ -neighborhood of M in the Gromov-Hausdorff distance.

Remark 1.3. Theorem 1.2 is new even in the case of M being a Riemannian manifold. Together with Theorem 1.1, it enables us to compute the homotopy type of M in terms of only the covering data of a good covering having only finite types.

In the course of the proofs of Theorems 1.1 and 1.2, we obtain the following, which is needed in the proof of Theorem 1.1(1) to show the conical property in the conditions of good coverings.

Theorem 1.4. *Let $f : U \rightarrow \mathbb{R}$ be a proper strictly concave function defined on a connected open subset U of an Alexandrov space M . Then*

- (1) *there is a point $p \in U$ such that $\Omega := \{f \geq a\}$ is convex SLC to p for any a with $\inf_U f < a < \max_U f$.*
- (2) *Ω is conical if either Ω does not meet ∂M , or Ω meets ∂M and the function $\tilde{f} : D(\Omega) \rightarrow \mathbb{R}$ naturally induced by f on the double $D(\Omega)$ of Ω is strictly concave.*

Here the double $D(\Omega)$ is defined as the disjoint union $\Omega \amalg \Omega$ glued along their boundaries $\Omega \cap \partial M$.

In Theorem 1.4(2), we have counter examples if we drop the assumption on \tilde{f} . It should also be remarked that in Theorem 1.4, the gradient

flow of a strictly concave function f might take infinite time to reach the unique maximum point of f in general. Therefore the gradient flow of f is not enough for the construction of a strong Lipschitz contraction, and we need additional arguments in the proof.

In the proof of Theorem 1.1(1), we also need to establish a version of Perelman's fibration theorem:

Theorem 1.5 (cf. [16], [17], [18]). *Let $f : U \rightarrow \mathbb{R}$ be a proper semiconcave function defined on an open set U of an Alexandrov space. If one of the following conditions holds, then f is a fiber bundle over $f(U)$:*

- (1) *if U does not meet ∂M , f is regular on U ,*
- (2) *if U meets ∂M , the canonical extension of f to the double of U is also semiconcave and is regular on it.*

Theorem 1.5 was proved for admissible functions f on U possibly with boundary in [16] and [17], for semiconcave functions on U without boundary in [18]. Our contribution is in the case when f is a semiconcave function and U meets ∂M .

The organization of the paper is as follows. In Section 2, we briefly recall several notions about Alexandrov spaces, SLC neighborhoods and semiconcave functions. In Section 3, we prove that a metric ball is SLC if the distance function from the center is regular on the ball. This extends a previous result in [14]. To achieve this, we develop a consecutive gluing method of gradient flows of several distance functions by proving the Lipschitz regularity of an implicit function. Such a gluing procedure is turned out to be useful also in the proof of Theorem 1.4. In Sections 4 and 5, we prove Theorems 1.1 and 1.2 respectively by making use of Theorem 1.4. Theorem 1.5 is proved in Section 6.

2. PRELIMINARIES

Let us recall the definition of Alexandrov spaces and related fundamental facts. For more details, we refer to [3] and [2]. Throughout the present paper, we denote by $|xy|$ the distance between points x and y in a metric space.

2.1. Basics of Alexandrov spaces. A metric space is said to be geodesic if any two points in the space can be joined by a minimal geodesic, where a minimal geodesic is an isometric embedding from an interval.

We say that a geodesic complete metric space M is an *Alexandrov space* (of curvature bounded locally from below) if for each $p \in M$, there exist $r > 0$ and $\kappa \in \mathbb{R}$ such that for any distinct four points $a_i \in B(p, r)$, $i = 0, 1, 2, 3$ with $\max_{1 \leq i < j \leq 3} \{|a_0a_i| + |a_0a_j| + |a_ia_j|\} < \pi/\sqrt{\kappa}$ if $\kappa > 0$, we have

$$\sum_{1 \leq i < j \leq 3} \tilde{\angle}_\kappa a_i a_0 a_j \leq 2\pi.$$

Here, $\tilde{\angle}_{\kappa}bac$ denotes the inner angle of a geodesic triangle of length $|ab|$, $|bc|$ and $|ca|$, at the vertex with opposite side of length $|bc|$, in a simply-connected complete surface of curvature κ . In the present paper, we only deal with finite-dimensional Alexandrov spaces.

From now on, let M denote an n -dimensional Alexandrov space. For an Alexandrov space $M = (M, |\cdot, \cdot|)$ and $r > 0$, we denote by rM the space $(M, r|\cdot, \cdot|)$. For $p \in M$, the pointed Gromov-Hausdorff limit of (rM, p) as $r \rightarrow \infty$ always exists and is denoted by $(T_p M, o)$, which is called the *tangent cone* of M at p . An element of $T_p M$ is called a vector. For two vectors $v, w \in T_p M$, we set $\langle v, w \rangle = |v||w| \cos \angle_{vow}$ if $|v| \neq 0 \neq |w|$ and $\langle v, w \rangle = 0$ otherwise, where $|v|$ is the distance from v to the origin o .

For $p \in M$, the set of all non-trivial unit-speed geodesic starting at p is denoted by Σ'_p , which admits an equivalent relation defined by $\gamma \sim \sigma$ if and only if $\angle(\gamma, \sigma) = \lim_{s,t \rightarrow 0} \tilde{\angle}_{\kappa} \gamma(s)p\sigma(t) = 0$, for fixed κ . The equivalent class of γ is denoted by $\gamma^+(0)$, where γ is assumed to be parametrized $\gamma(0) = p$. Then, \angle is a metric on the set of all equivalent classes. The completion of it by \angle is denoted by Σ_p and is called the *space of directions* at p . Each element of Σ_p is called a direction. For $q \neq p$, we denote by $\uparrow_p^q \in \Sigma_p$ the direction of a minimal geodesic from p to q at p .

The tangent cone $T_p M$ is isometric to the Euclidean cone over Σ_p . So, any vector $v \in T_p M$ can be written as $v = a\xi$ for some $a \geq 0$ and $\xi \in \Sigma_p$.

For a Lipschitz curve $c : [0, a] \rightarrow M$, it has the direction at $t = 0$ if $\overline{\lim}_{s,t \rightarrow 0} \tilde{\angle}_{\kappa} c(s)c(0)c(t) = 0$ holds, for some fixed κ . Then, the vector $c^+(0)$ is canonically defined as the limit of $|c(0)c(t)| \uparrow_{c(0)}^{c(t)}$ as $t \rightarrow 0$.

The boundary ∂M is defined as the set of all points $p \in M$ such that Σ_p has non-empty boundary. Here, one-dimensional Alexandrov spaces are one-dimensional Riemannian manifolds possibly with boundary, whose boundaries are defined as the boundaries of manifolds.

2.2. Strong Lipschitz contractibility.

Definition 2.1 ([14]). Let X be a metric space, $p \in X$ and $r > 0$. We say that a subset Ω of X is *strongly Lipschitz contractible* (abbreviated by SLC) to some point $p \in \Omega$, if there is a map

$$H : \Omega \times [0, 1] \rightarrow \Omega$$

which is Lipschitz in the sense that

$$|H(x, s)H(y, t)| \leq A|xy| + B|s - t|$$

holds on the domain for some $A, B \geq 0$, such that $H_0(x) = x$, $H_1(x) = p$, and the distance

$$d(H_t(x), p)$$

is monotone non-increasing in t for every $x \in \Omega$. Here, $H_t(x) = H(x, t)$.

For a subset $A \subset \Omega$, we say that Ω is *strongly Lipschitz contractible to A* if there is a Lipschitz map $H : \Omega \times [0, 1] \rightarrow \Omega$ such that $H_0(x) = x$ and $H_1(x) \in A$ for every $x \in \Omega$, the function $d(H_t(y), A)$ is monotone non-increasing in t for every $y \in \Omega$ and $H_t(z) = z$ for all $z \in A$ and $t \in [0, 1]$.

Note that if $B(p, r)$ is SLC to p , then $B(p, r')$ is also SLC to p for every $r' < r$. Here, $B(q, s)$ denotes the closed metric ball centered at q of radius s .

In [14], we proved that every Alexandrov space is *strongly locally Lipschitz contractible* in the following sense.

Theorem 2.2 ([14]). *Let M be an Alexandrov space. For every $p \in M$, there is an $r > 0$ such that $B(p, r)$ is strongly Lipschitz contractible to p . Here, $B(p, r)$ denotes the closed metric ball centered at p of radius r .*

2.3. Semiconcave functions and their gradient flows. Following [22], we recall the notion of the gradients of semiconcave functions on Alexandrov spaces and their properties.

Let M be an Alexandrov space. A locally Lipschitz function f defined on an open subset U of M is said to be *semiconcave* if for any $x \in U$, there are $r > 0$ and $\lambda \in \mathbb{R}$ such that for any minimal geodesic $\gamma : [0, T] \rightarrow U(x, r)$ of unit speed contained in $U(x, r)$, the function $f \circ \gamma(t) - (\lambda/2)t^2$ is concave on $(0, T)$ in the usual sense. In this case, f is said to be λ -concave at x and on $U(x, r)$. Let us set $\underline{\lambda}(x) = \inf\{\lambda \mid f \text{ is } \lambda\text{-concave at } x\}$. Then, $\underline{\lambda}$ is upper semicontinuous on U . Indeed, for any $\epsilon > 0$, there is $r > 0$ such that f is $(\underline{\lambda}(x) + \epsilon)$ -concave on $U(x, r)$. Then, f is $(\underline{\lambda}(x) + \epsilon)$ -concave on $U_y(r - |xy|)$. Hence, we have $\lim_{y \rightarrow x} \underline{\lambda}(y) \leq \underline{\lambda}(x)$. If a function $g : U \rightarrow \mathbb{R}$ satisfies $g(x) \geq \underline{\lambda}(x)$, we also say that f is g -concave. We say that f is *strictly concave* (concave, resp.) if $\underline{\lambda} < 0$ (≤ 0 , resp.) on the domain.

The distance function from a closed set A of an Alexandrov space M is semiconcave on $M \setminus A$.

Let f be a semiconcave function defined on an open subset U of an Alexandrov space M . For $x \in U$, we can define the *differential* $f' = f'_x : T_x M \rightarrow \mathbb{R}$ of f at x by

$$f'_x(c^+(0)) = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(c(0))}{t}$$

for any curve $c : [0, a) \rightarrow U$ with $c(0) = x$ having the direction at $t = 0$. The map $f'_x : T_x M \rightarrow \mathbb{R}$ is a 0-concave function.

The *gradient* of f at x is the vector $\nabla_x f = \nabla f \in T_x M$ uniquely determined by the relations

$$|\nabla f|^2 = f'(\nabla f) \text{ and } \langle \nabla f, v \rangle \geq f'(v)$$

for every $v \in T_x M$. The *gradient curve* of f is a curve $c : [0, a) \rightarrow M$ which has the direction at any time $t \in [0, a)$ and satisfies

$$c^+(t) = \nabla_{c(t)} f$$

for every $t \in [0, a)$.

Theorem 2.3 ([19], [21]). *For any semiconcave function f on an open subset U , and for any $x \in U$, there exists the unique maximal gradient curve starting at x .*

Let us recall a contraction property of gradient flows.

Lemma 2.4. *Let c_1, c_2 be two gradient curves of a λ -concave function f defined on U . Suppose that $c_1(t)$ and $c_2(t)$ can be joined by a minimal geodesic contained in U , for every t with $t_1 \geq t \geq t_0$. Then, we have $|c_1(t)c_2(t)| \leq e^{\lambda(t-t_0)}|c_1(t_0)c_2(t_0)|$ for $t_1 \geq t \geq t_0$.*

Proof. We may assume that $t_0 = 0$ and set $x_1 = c_1(0)$ and $x_2 = c_2(0)$. Let $\gamma : [0, |x_1 x_2|] \rightarrow U$ be a geodesic with $\gamma(0) = x_1$ and $\gamma(|x_1 x_2|) = x_2$. Since f is λ -concave along γ , we have

$$\frac{f(x_i) - f(x_j) - \frac{\lambda}{2}|x_1 x_2|^2}{|x_1 x_2|} \leq f'(\uparrow_{x_j}^y) \leq \langle \nabla f, \uparrow_{x_j}^y \rangle$$

for $(i, j) = (1, 2), (2, 1)$, where y is the midpoint in γ .

On the other hands, we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0+} |c_1(t)c_2(t)| &\leq (d_y)'_{x_1}(c_1^+(0)) + (d_y)'_{x_2}(c_2^+(0)) \\ &\leq -\langle \uparrow_{x_1}^y, \nabla f \rangle - \langle \uparrow_{x_2}^y, \nabla f \rangle \\ &\leq \lambda|x_1 x_2|. \end{aligned}$$

This immediately implies the assertion. \square

Let us recall the definition of polar vectors.

Definition 2.5 ([22]). Let C be a Euclidean cone of nonnegative curvature. For a vector $u \in C$ and a subset $\mathcal{V} \subset C$, we say that u is *polar to \mathcal{V}* if

$$\langle u, w \rangle + \sup_{v \in \mathcal{V}} \langle v, w \rangle \geq 0$$

for any $w \in C$.

Note that u is polar to \mathcal{V} if and only if

$$(2.1) \quad \phi(u) + \inf_{v \in \mathcal{V}} \phi(v) \leq 0$$

holds for any concave function $\phi : C \rightarrow \mathbb{R}$. A geometric meaning of vector being polar is explained as follows. For vectors $v, w \in C$, if $|v| = |w|$, then v is polar to w if and only if $\angle voz + \angle woz \leq \pi$ for any $z \in C$. So, if the space of directions at the origin of C has diameter not greater than $\pi/2$, any two vectors of the same length are polar. If C

isometrically splits as $C = C' \times \mathbb{R}$, then (o, t) is polar to $(o, -t)$, where $o \in C'$ is the origin and $t > 0$.

Lemma 2.6 ([22]). For a point p in an Alexandrov space M and a closed subset A of M with $p \notin A$, the gradient $\nabla_p d_A$ is polar to A'_p . Here, A'_p is the set of all directions of a minimal geodesic from p to A .

Proof. Let us fix $w \in T_p M$. Let $\xi \in A'_p$ be a direction such that $(d_A)'(w) = -\max_{\eta \in A'_p} \langle \eta, w \rangle = -\langle \xi, w \rangle$. Then, we have

$$\langle \nabla d_A, w \rangle + \langle \xi, w \rangle \geq 0.$$

This completes the proof. \square

3. STRONGLY LIPSCHITZ CONTRACTIBLE BALLS

In this section, we prove

Theorem 3.1. *Let p be a point in an Alexandrov space and $r > 0$. If d_p is regular on $B(p, r) \setminus \{p\}$, then $B(p, r)$ is strongly Lipschitz contractible to p .*

This is a global version of Theorem 2.2. To prove Theorem 3.1, we prove

Theorem 3.2. *Let f be a proper semiconcave function defined on an open set U such that f is regular on $f^{-1}[a, b]$ for some $a < b$. Then, there is a Lipschitz map $H : \{f \leq b\} \times [0, 1] \rightarrow \{f \leq b\}$ such that for every $x \in f^{-1}[a, b]$, $y \in \{f \leq a\}$ and $t \in [0, 1]$, we have*

- $H_0(x) = x$, $f(H_1(x)) = a$;
- the function $f(H_t(x))$ is monotone non-increasing in t ;
- $H_t(y) = y$.

This theorem is proved in §3.3. Remark that for an f as in Theorem 3.2, the gradient flow of f increases the value of f . Since Theorem 3.2 gives a “reverse flow” of it in some sense, the existence of such a flow is non-trivial. Such a reverse flow is important for applications.

Proposition 3.3. *Let f be a semiconcave proper function defined on an open subset U in an Alexandrov space M . Suppose that there is an $r \in \mathbb{R}$ such that f is regular on $f^{-1}(r)$. Then, there exist $r', r'', \bar{r} \in f(U)$ with $r' < r < r'' < \bar{r}$ such that the distance function $d_{f^{-1}(\bar{r})}$ from the level set $f^{-1}(\bar{r})$ is regular on $f^{-1}[r', r''] \subset U$. Further,*

$$f'(\nabla d_{f^{-1}(\bar{r})}) < -c$$

holds on $f^{-1}[r', r'']$, for some $c > 0$.

Proof. Since $f^{-1}(r)$ is compact and f is regular on $f^{-1}(r)$, by the lower semicontinuity of the absolute gradient, $|\nabla f| > c$ on $f^{-1}(r)$ for some $c > 0$. Let λ be such that f is λ -concave near $f^{-1}(r)$. We may assume that $\lambda \geq 0$. Let $\nu > 0$ be taken so that for any $x \in f^{-1}(r)$ and $y \in U$

with $|xy| < \nu$, every minimal geodesic between them is contained in U . For instance, we set ν the half of $|f^{-1}(r), M \setminus U|$.

First, we prove that there are $\delta > 0$ and $\bar{\ell} > 0$ such that for any $x \in f^{-1}[r - \delta, r + \delta]$, there is $y \in U$ with $\nu > |xy| > \bar{\ell}$ and

$$\frac{f(y) - f(x) - \frac{\lambda}{2}|xy|^2}{|xy|} > c.$$

Using it, we completes the proof of the lemma.

By the assumption, for any $x \in f^{-1}(r)$, there exists $y \in U$ with $|yx| < \nu$ such that

$$\frac{f(y) - f(x) - \frac{\lambda}{2}|yx|^2}{|yx|} > c.$$

Fixing x and y , there is $\epsilon > 0$ such that if $z \in B(x, \epsilon)$, then $|zy| < \nu$ and

$$\frac{f(y) - f(z) - \frac{\lambda}{2}|yz|^2}{|yz|} > c.$$

Since $f^{-1}(r)$ is compact, there are finitely many points $x_1, \dots, x_m \in f^{-1}(r)$, $y_1, \dots, y_m \in U$ and positive numbers $\epsilon_1, \dots, \epsilon_m$ such that $f^{-1}(r) \subset \bigcup_{1 \leq i \leq m} B(x_i, \epsilon_i) \subset U$ and that if $x \in B(x_i, \epsilon_i)$, then $|xy_i| < \nu$ and

$$\frac{f(y_i) - f(x) - \frac{\lambda}{2}|y_i x|^2}{|y_i x|} > c.$$

There is $\delta_0 > 0$ such that $f^{-1}[r - \delta_0, r + \delta_0] \subset \bigcup_{1 \leq i \leq m} B(x_i, \epsilon_i)$.

Now, for each $x \in f^{-1}[r - \delta, r + \delta]$, let us define the value $\ell(x)$ as follows. Setting $L_x = \left\{ y \in U \mid |xy| < \nu \text{ and } \frac{f(y) - f(x) - \frac{\lambda}{2}|xy|^2}{|xy|} > c \right\}$ and $\ell(x) = \sup\{|xy| \mid y \in L_x\}$. Obviously, the function $x \mapsto \ell(x)$ is lower semicontinuous. We set $\ell_\delta := \min\{\ell(x) \mid x \in f^{-1}[r - \delta, r + \delta]\}$ for $0 \leq \delta \leq \delta_0$. Then, ℓ_δ converges to ℓ_0 as $\delta \rightarrow 0$. Since $\ell_0 > 0$, some $\delta > 0$ exists so that

$$r - \delta + c\ell_\delta > r + \delta.$$

We fix some constant \bar{r} with $r - \delta + c\ell_\delta > \bar{r} > r + \delta$, and define $\bar{\ell} > 0$ by $\bar{r} = r - \delta + c\bar{\ell}$.

Then, for any $x \in f^{-1}[r - \delta, r + \delta]$, there is $y \in U$ with $\bar{\ell} < |xy| < \nu$ and

$$\frac{f(y) - f(x) - \frac{\lambda}{2}|yx|^2}{|yx|} > c.$$

Note that $f(y) > f(x) + c|xy| > \bar{r} > r + \delta \geq f(x)$. So, there is a point z in a geodesic between x and y such that $f(z) = \bar{r}$. By the λ -concavity of f , we obtain

$$\frac{f(z) - f(x) - \frac{\lambda}{2}|zx|^2}{|zx|} \geq \frac{f(y) - f(x) - \frac{\lambda}{2}|yx|^2}{|yx|} > c.$$

Let $w \in f^{-1}(\bar{r})$ be a point so that $|xw| = \min\{|x\bar{w}| \mid \bar{w} \in f^{-1}(\bar{r})\}$. Then, we have

$$f'_x(\uparrow_x^w) + \frac{\lambda}{2}|wx| \geq \frac{f(w) - f(x)}{|wx|} \geq \frac{f(z) - f(x)}{|zx|} > c + \frac{\lambda}{2}|zx|.$$

Hence, $f'_x(\uparrow_x^w) > c$. Since $\nabla_x d_{f^{-1}(\bar{r})}$ is polar to $f^{-1}(\bar{r})'_x \subset \Sigma_x$, by (2.1), we obtain

$$f'_x(\nabla d_{f^{-1}(\bar{r})}) < -c.$$

This completes the proof. \square

Proposition 3.3 enables us to check that the gradient flow of the distance function from $f^{-1}(\bar{r})$ makes a Lipschitz flow whose flow curves decrease the value of f . When the curves arrive at the level set $f^{-1}(r')$, we use Proposition 3.3 again and obtain the gradient flow of the distance function from some level set $f^{-1}(r' + \epsilon)$ for some $\epsilon > 0$. Then, we connect two flows on $f^{-1}(r')$ and that check that the obtained flow is also Lipschitz, in the next two subsections.

3.1. Lipschitz regularity of an implicit function. Let f be a proper semiconcave function defined on an open set U which is regular on U . Let Φ denote the maximal gradient flow of f . For $x \in U$, the maximal time defining the flow $\Phi(x, \cdot)$ on U is denoted by T_x . We assume that there are a proper semiconcave function g defined on U , real numbers $a < b$ and $c > 0$ such that $g(U) \supset [a, b]$ and that

$$(3.2) \quad g'_x(\nabla_x f) < -c$$

for every x in a neighborhood of $g^{-1}[a, b]$. Further, we assume that for some $\bar{a} < a$ and $b < \bar{b}$, we may assume that $g'_x(\nabla_x f) < -c$ on $g^{-1}[\bar{a}, \bar{b}]$. In particular, $g(\Phi(x, t))$ is strictly decreasing in t whenever $\Phi(x, t) \in g^{-1}[\bar{a}, \bar{b}]$. For any $x \in g^{-1}[a, b]$, we define the first hitting time to $\{g \leq a\}$ of x by

$$t(x) := \min\{t \in [0, T_x) \mid \Phi(x, t) \in \{g \leq a\}\}.$$

The condition (3.2) implies that the set of all t 's with $\Phi(x, t) \in \{g \leq a\}$ has the form $[t(x), T_x)$. Further, $g(\Phi(x, t)) = a$ if and only if $t = t(x)$. Then, $x \mapsto t(x)$ can be checked to be continuous. We also easily check that some T exists so that $t(x) \leq T$ for all $x \in g^{-1}[a, b]$. For instance, we set $T = (b - a)/c$.

If $\epsilon > 0$ is taken to be so small, then we have that for any $x, y \in g^{-1}[a, b]$ with $|xy| < \epsilon$, every minimal geodesic segment between x and y is contained in $g^{-1}[\bar{a}, \bar{b}]$. Indeed, we take ϵ as a positive number smaller than $\min\{a - \bar{a}, \bar{b} - b\}/2\text{Lip}(g)$.

Lemma 3.4 (Implicit function lemma). *Let $f, g, U, a, b, \bar{a}, \bar{b}, \epsilon$ be as above. Then, the function $g^{-1}[a, b] \ni x \mapsto t(x) \in [0, T]$ is Lipschitz continuous. Further, if $x, y \in g^{-1}[a, b]$ with $|xy| < \epsilon$, then we have*

$$|t(x) - t(y)| \leq L(f, g, c, a, b, \epsilon)|xy|.$$

for some constant $L(f, g, c, a, b, \epsilon)$ depending on f, g, c, a, b, ϵ .

Proof. For $x \in g^{-1}[a, b]$, if $\Phi(x, t) \in g^{-1}[\bar{a}, \bar{b}]$, then we have

$$\left(\frac{d}{dt} \right)_+ g(\Phi(x, t)) = g'_{\Phi(x, t)}(\nabla f) < -c.$$

Let λ be a constant so that f is λ -concave on $g^{-1}[a, b]$. Then, $\text{Lip}(\Phi(\cdot, t)) \leq e^{\lambda t}$ on U . If a geodesic segment γ of constant speed is contained in $g^{-1}[\bar{a}, \bar{b}]$, then the function $g(\Phi(\gamma(s), t))$ is Lipschitz in s , so it has the derivative for almost all s with

$$\left| \frac{d}{ds} g(\Phi(\gamma(s), t)) \right| \leq \text{Lip}(g) e^{\lambda t} |\dot{\gamma}(s)|.$$

Let us take points $x, y \in g^{-1}[a, b]$ with $|xy| < \epsilon$ and a geodesic segment $\gamma : [0, 1] \rightarrow g^{-1}[\bar{a}, \bar{b}]$ of constant speed $|\dot{\gamma}(s)| \equiv |xy|$ with $\gamma(0) = x$ and $\gamma(1) = y$. We assume that $t(y) > t(x)$. Then, setting $\sigma(s) = t(x) + s(t(y) - t(x))$, we have

$$\begin{aligned} 0 &= g(\Phi(y, t(y))) - g(\Phi(x, t(x))) \\ &= \int_0^1 \frac{d}{ds} g(\Phi(\gamma(s), \sigma(s))) ds \\ &= \int_0^1 \frac{d}{ds} \Big|_{s_1=s} g(\Phi(\gamma(s_1), \sigma(s))) + \frac{d}{ds} \Big|_{s_2=s} g(\Phi(\gamma(s), \sigma(s_2))) ds \end{aligned}$$

Now, we have

$$\begin{aligned} \frac{d}{ds} \Big|_{s_2=s} g(\Phi(\gamma(s), \sigma(s_2))) &= g'(\nabla f) \dot{\sigma}(s) = g'(\nabla f)(t(y) - t(x)) \\ &< -c(t(y) - t(x)) \end{aligned}$$

Hence, we obtain

$$\begin{aligned} c(t(y) - t(x)) &< \int_0^1 \frac{d}{ds} \Big|_{s_1=s} g(\Phi(\gamma(s_1), \sigma(s))) ds \\ &\leq \int_0^1 \left| \frac{d}{ds} \Big|_{s_1=s} g(\Phi(\gamma(s_1), \sigma(s))) \right| ds \\ &\leq \text{Lip}(g) e^{\lambda T} |\dot{\gamma}(s)| = \text{Lip}(g) e^{\lambda T} |xy| \end{aligned}$$

This provides the second assertion in the conclusion. Since $g^{-1}[a, b]$ is compact, $t(\cdot)$ is Lipschitz on $g^{-1}[a, b]$. This completes the proof. \square

3.2. Gluing two gradient flows. Let U be a bounded open subset of an Alexandrov space, and g, h a semiconcave function defined on U . Let f be a semiconcave function defined on an open set V with

$V \subset U$. Suppose that there exist a, b, c, d with $a < b < c < d$ so that $\{h \leq c\} \subset V$, $\{h \leq d\} \subset U$ and that

$$\begin{aligned} h'(\nabla f) &< -A \text{ on a neighborhood of } h^{-1}[a, c], \\ h'(\nabla g) &< -A \text{ on a neighborhood of } h^{-1}[b, d] \end{aligned}$$

for some constant $A > 0$. Let Φ and Ψ denote the gradient flows of f and g , respectively.

Lemma 3.5 (Gluing lemma). *Let $U, V, f, g, h, a, b, c, d, \Phi, \Psi$ be as above. Then, there exists a locally Lipschitz map $H : \{h \leq d\} \times [0, \infty) \rightarrow \{h \leq d\}$ such that*

- $H(x, 0) = x$ for $x \in \{h \leq d\}$;
- $H(x, t) = \Psi(x, t)$ for $(x, t) \in h^{-1}[c, d] \times [0, \epsilon]$;
- $H(x, t) = \Phi(x, t - T)$ for $x \in \{h \leq b\}$ and $t \in [T, T_x]$,

for some $T > 0$ and $\epsilon > 0$. Further, the function $h(H(x, t))$ is monotone non-increasing in t for every $x \in \{h \leq d\}$.

Proof. For any $y \in h^{-1}[b, d]$, we set $t(y) = \min\{t \in [0, \infty) \mid \Psi(y, t) \in \{h \leq b\}\}$. By Lemma 3.4, the function $h^{-1}[b, d] \ni y \mapsto t(y)$ is Lipschitz. Let $T = \max\{t(y) \mid y \in h^{-1}[b, d]\} \leq (d - b)/A$. Let us define the map $H : \{h \leq d\} \times [0, \infty) \rightarrow \{h \leq d\}$ by

$$H(x, t) = \begin{cases} \Psi(x, t) & \text{if } x \in h^{-1}[b, d], t \leq t(x) \\ \Psi(x, t(x)) & \text{if } x \in h^{-1}[b, d], t \in [t(x), T] \\ \Phi(\Psi(x, t(x)), t - T) & \text{if } x \in h^{-1}[b, d], t \geq T \\ x & \text{if } x \in \{h \leq b\}, t \in [0, T] \\ \Phi(x, t - T) & \text{if } x \in \{h \leq b\}, t \geq T. \end{cases}$$

Let us set $\epsilon = \min\{t(y) \mid y \in h^{-1}[c, d]\} > 0$. The map H satisfies that $H(y, t) = \Psi(y, t)$ for $y \in h^{-1}[c, d]$ and $t \leq \epsilon$. \square

3.3. Proof of Theorems 3.1 and 3.2. Let us first prove Theorem 3.2.

Proof of Theorem 3.2. Let $f : U \rightarrow \mathbb{R}$ be a proper semiconcave function which is regular on $f^{-1}[a, b]$ for some $a < b$. Let $c > 0$ be a number satisfying $|\nabla f| > c$ on $f^{-1}[a, b]$. By Proposition 3.3, there are $a = t_0 < t_1 < \dots < t_N = b$ with finite sequences of positive numbers $\{\tau_i\}_{i=1}^N$ and $\{\delta_i\}_{i=1}^N$ such that $\delta_i < \tau_i$ and

$$f'(\nabla_x d_{f^{-1}(t_i + \tau_i)}) < -c$$

on $f^{-1}[t_i - \delta_i, t_i + \delta_i]$, and that $\bigcup_{i=1}^N (t_i - \delta_i, t_i + \delta_i) \supset [a, b]$. Using Lemma 3.5 repeatedly, we obtain a Lipschitz map

$$H : \{f \leq b\} \times [0, T] \rightarrow \{f \leq b\}$$

such that $H_0(x) = x$, $H_T(x) \in \{f = a\}$ and $f(H_t(x))$ is monotone non-increasing in t for every $x \in \{f \leq b\}$. Further, $H(x, t)$ coincides

with the gradient flow $\Phi(x, t - A)$ of $d_{f^{-1}(t_0 + \tau_0)}$ with some parameter translation A , if t is close to T and $f(x)$ is close to a . For $x \in \{f \leq b\}$, we set $t(x) := \min\{t \geq 0 \mid H(x, t) \in \{f \leq a\}\}$. Then, by Lemma 3.4, the map $t(\cdot)$ is Lipschitz.... Let us define $G : \{f \leq b\} \times [0, T] \rightarrow \{f \leq b\}$ by

$$G(x, t) := \begin{cases} H(x, t) & \text{if } t \leq t(x) \\ H(x, t(x)) & \text{if } t \geq t(x). \end{cases}$$

Then, G is Lipschitz such that $G(x, 0) = x$ and $G(x, T) \in \{f = a\}$ for all $x \in \{f \leq b\}$ and $G(y, t) = y$ for all $y \in \{f \leq a\}$ and $t \in [0, T]$. Further, $f(G(x, t))$ is monotone non-increasing in t , for every $x \in f^{-1}[a, b]$. This completes the proof. \square

Remark 3.6. Remark that if a semiconcave function f is globally defined on a compact Alexandrov space X and has the following gradient estimate

$$(d_p)'(\nabla_x f) < -c$$

for every $x \in X \setminus \{p\}$, for some $p \in M$ and a uniform constant $c > 0$, then the gradient flow Φ of f can reach p in a uniform finite time. Hence, up to time scaling, Φ gives a strong Lipschitz contraction from X to p .

If no such a gradient estimate of a semiconcave function exists, then its gradient flow may not give a strong Lipschitz contraction. Indeed, there is a strictly concave function such that the gradient flow does not reach its unique critical point in any finite time.

Let us consider a strictly concave function $f(x) = -x^2/2$ on $[-1, 1]$. The unique critical point is the zero 0. For $x \in (0, 1]$, the gradient of f at x is determined as follows.

$$|\nabla f|_x = x \text{ and } \nabla f(x) = x \uparrow_x^0.$$

Let us consider the gradient curve $\alpha : [0, \infty) \rightarrow [-1, 1]$ of f with $\alpha(0) = 1$. In this case, it is determined by the following single differential equation.

$$|\alpha^+(t)| = |\nabla f|_{\alpha(t)}.$$

Since

$$|\alpha^+(t)| = \lim_{\delta \rightarrow 0+} \frac{|\alpha(t + \delta) - \alpha(t)|}{\delta} = \lim_{\delta \rightarrow 0+} \frac{\alpha(t) - \alpha(t + \delta)}{\delta},$$

we have

$$\alpha(t) = e^{-t}.$$

Therefore, the curve $\alpha(t)$ does not reach 0 in finite time.

Proof of Theorem 3.1. Let p be a point in an Alexandrov space M and $r > 0$ such that d_p is regular on $B(p, r) \setminus \{p\}$. By Theorem 2.2, there is $r_0 > 0$ and a strong Lipschitz contraction F from $B(p, r_0)$ to p . So,

we may assume that $r_0 < r$. Then, by the assumption, d_p is regular on $d_p^{-1}[r_0, r]$. From Theorem 3.2, there is a Lipschitz map

$$H : B(p, r) \times [0, 1] \rightarrow B(p, r)$$

such that $H(x, 0) = x$, $H(x, 1) \in S(p, r_0)$ and $d_p(H(x, t))$ is monotone non-increasing in t for every $x \in B(p, r_0)$ and that $H(y, t) = y$ for every $(y, t) \in B(p, r_0) \times [0, 1]$. Gluing two homotopies F and H in a natural way, we obtain a strong Lipschitz contraction from $B(p, r)$ to p . This completes the proof. \square

4. EXISTENCE OF GOOD COVERING AND HOMOTOPY TYPES

Let M be an Alexandrov space. A subset of M is called a *domain* if it is a connected open subset. A domain U of M is *conical* if there are $x \in U$ and a topological space A such that (U, x) is homeomorphic to the open cone $(K(A), o)$ as a pointed space. Then, U is called a conical neighborhood of x and A is called a generator of U . From the uniqueness of conical neighborhoods ([12]) and Perelman's stability theorem ([16], [11]), (U, x) must be homeomorphic to $(K(\Sigma_x), o)$. So, the generator A is compact and has the same (co)homology groups as those of Σ_x . However note that A is not homotopic to Σ_x , in general. For instance, if M is the cone over the suspension of a homology sphere X , then the apex o of M has a conical neighborhood with generator homeomorphic to a sphere, however $\Sigma_o = \Sigma(X)$ is not homeomorphic to a sphere.

A subset V of M is called *convex* if every minimal geodesic segment joining any two points of V is contained in V . Here it should be noted that the uniqueness of geodesics does not hold in general as shown by the double of a flat disk.

To prove Theorem 1.1, we need

Theorem 4.1 (cf. [17], [10], [13]). *For any p in an Alexandrov space M , there exist an open neighborhood Ω of p and a strictly concave function f defined on Ω such that*

- (1) $f(p) = \max_{\Omega} f$;
- (2) $\{f > c\}$ is convex and conical SLC to p for any c with $\inf_{\Omega} f < c \leq \max_{\Omega} f$.

First it should be noted that the strictly concave function f in Theorem 4.1 was constructed in [17], [10] as the minimum of the average of the composition of distance functions and a strictly concave C^2 -function, by using some net in a metric sphere around p . More explicitly this is done as follows : Let $r > 0$ be small enough. Fixing some maximal ℓ -discrete set $\{x_\alpha\}_\alpha$ of $S(p, 2r)$, a maximal ν -discrete set $\{x_{\alpha\beta}\}_\beta$ of $S(p, 2r) \cap B(x_\alpha, 2\ell r)$ with $\nu \ll \ell$ and a concave increasing function $\chi : (0, 3r) \rightarrow \mathbb{R}$ which is strictly concave near r , we set $f_\alpha = \frac{1}{\#\{\beta\}} \sum_\beta \chi(d(x_{\alpha\beta}, \cdot))$ and $f = \min_\alpha f_\alpha$. Then f is strictly concave

and regular on $U(p, r)$ except p . It is checked that the set $\{f > c\}$ is SLC to p for some c with $c < \max_{U(p, r)} f$ (see [13]). We only have show that $\{f > c\}$ is conical. If $p \notin \partial M$, Theorem 1.5 implies the conclusion. If $p \in \partial M$, we take the metric ball $\tilde{B}(p, 3r)$ in the double $D(M)$, and take an ℓ -discrete set $\{\tilde{x}_\alpha\}_\alpha$ of $\tilde{S}(p, 2r) := \partial \tilde{B}(p, 2r)$, a maximal ν -discrete set $\{\tilde{x}_{\alpha\beta}\}_{\beta}$ of $\tilde{S}(p, 2r) \cap \tilde{B}(x_\alpha, 2\ell r)$ in such a way that those are invariant under the action of reflection with respect to ∂M . Then the function $\tilde{f} : \tilde{U}(p, r) \rightarrow \mathbb{R}$ defined by the distance functions from those points in a similar way to the above is strictly concave and regular except p . Thus by Theorem 1.5 f is a fiber bundle when restricted to $\{\max f > f > c\}$, and hence $\{f > c\}$ must be conical.

Proof of Theorem 1.4. Take a with $\inf_U f < a < \max_U f$, and let $\Omega := \{f > a\}$. Since f is strictly concave, a maximizer of it is unique, say $p \in \Omega$. Then, for any $x \in \Omega \setminus \{p\}$, we have from the concavity of f

$$(4.3) \quad f'(\uparrow_x^p) \geq \frac{f(p) - f(x)}{|px|} > 0.$$

Therefore f is regular on $\Omega \setminus \{p\}$. Let us take $c > 0$ such that f is $(-c)$ -concave on Ω . For $x \neq p$, the $(-c)$ -concavity implies

$$f(x) \leq f(p) + f'(\uparrow_p^x)|px| - (c/2)|px|^2 \leq f(p) - (c/2)|px|^2.$$

Therefore, (4.3) is improved by

$$(4.4) \quad f'(\uparrow_x^p) \geq \frac{f(p) - f(x)}{|px|} \geq (c/2)|px|.$$

Hence, $f'(\uparrow_x^p)$ has a uniform lower bound on $\{|px| \geq r\}$ depending on r , for every fixed $r > 0$. By the first variation formula, we have

$$(d_p)'(\nabla_x f) \leq -\langle \uparrow_x^p, \nabla_x f \rangle \leq -f'(\uparrow_x^p).$$

This together with (4.4) implies

$$(d_p)'(\nabla_x f) \leq -(c/2)r.$$

for every $x \in \{d_p \geq r\}$. Take $r > 0$ with $B(x, r) \subset \Omega$. From Theorem 3.2, there is a Lipschitz homotopy

$$F : \Omega \times [0, 1] \rightarrow \Omega$$

such that $F(x, 0) = x$, $|p, F(x, 1)| = r$ and $|p, F(x, t)|$ is monotone non-increasing in t for every $x \in \{d_p \geq r\}$ and that $F(y, t) = y$ for every $(y, t) \in B(p, r) \times [0, 1]$.

On the other hands, Theorem 2.2 gives a strong Lipschitz contraction G from $B(p, r)$ to p , if r is small. Gluing two Lipschitz homotopies F and G in a natural way, we have a Lipschitz homotopy

$$H : \Omega \times [0, 1] \rightarrow \Omega$$

such that $H(x, 0) = x$, $H(x, 1) = p$ and $|p, H(x, t)|$ is monotone nonincreasing in t , for every $x \in \Omega$. This completes the proof of (1).

For the proof of (2), we only have to use Theorem 1.5. \square

Lemma 4.2. *Let U_1, \dots, U_m be convex, conical SLC domains in M defined as superlevel sets $U_i = \{f_i > c_i\}$ via strictly concave functions f_i as in Theorem 4.1 defined on domains Ω_i . If $U_1 \cap \dots \cap U_m$ is nonempty, it is a convex, conical SLC-domain.*

Proof. We may assume $U_i = \{f_i > 0\}$, where f_i is $(-c)$ -concave for some $c > 0$. Set $\Omega := \bigcap_{i=1}^m \Omega_i$. Then

$$U_1 \cap \dots \cap U_m = \{x \in \Omega \mid \min_{1 \leq i \leq m} f_i(x) > 0\}.$$

Since $\min_{1 \leq i \leq m} f_i$ is $(-c)$ -concave on Ω , the conclusion follows from Theorem 1.4 if Ω does not meet ∂M . In case Ω meets ∂M , we first construct \mathbb{Z}_2 -equivariant $(-c)$ -concave function \tilde{f}_i on the double $D(U_i)$ in a way similar to the construction right after Theorem 4.1. Therefore \tilde{f}_i descends to a $(-c)$ -concave function f_i on U_i , and again we can apply Theorem 1.4 to get that the set $\bigcap_{1 \leq i \leq m} U_i$ is a convex, conical SLC domain. \square

Proof of Theorem 1.1(1). Let \mathcal{V} be an open covering of M . For any $x \in M$, we fix $V_x \in \mathcal{V}$ with $x \in V_x$. By Theorem 4.1, there is a strictly concave function f_x defined on some neighborhood Ω_x of x with $\Omega_x \subset V_x$. Adding a constant to f_x , we may assume $U_x = \{y \in \Omega_x \mid f_x(y) > 0\}$. By Lemma 4.2, U_x is a conical, convex SLC neighborhood of x . Since M is proper, it is covered by a countable union of compact subsets. Therefore we can choose a countable set $\{x_i\} \subset M$ such that $\{U_{x_i}\}$ is a locally finite covering of M . If the intersection $U_{x_1} \cap \dots \cap U_{x_m}$ is nonempty, we can set

$$U_{x_1} \cap \dots \cap U_{x_m} = \left\{ x \in \bigcap_{i=1}^m \Omega_{x_i} \mid \min_{1 \leq i \leq m} f_{x_i}(x) > 0 \right\},$$

it must be a convex, conical SLC domain by Lemma 4.2. This completes the proof. \square

Proof of Theorem 1.1(2). We consider only the case that M is noncompact. Let $\mathcal{U} = \{U_i\}_{i=1}^\infty$ be a locally finite good covering of M . Recall U_i is defined as the super level set

$$U_i = \{\varphi_i > 0\},$$

of a strictly concave function φ_i . Define $\psi_i = \varphi_i - c_i$ on U_i , and $\psi_i = 0$ outside U_i , and set

$$f_i = \frac{\psi_i}{\sum_{j=1}^\infty \psi_j}.$$

Note that f_i is a Lipschitz function on M satisfying

- (1) $0 \leq f_i \leq 1$;
- (2) $f_i > 0$ on U_i and $\text{supp}(f_i) = \bar{U}_i$;
- (3) $\sum_{i=1}^\infty f_i = 1$.

Let K be the nerve of the covering $\{U_i\}_{i=1}^\infty$ where the vertices of K are the canonical basis $\{e_i\}_{i=1}^\infty$ of \mathbb{R}^∞ . Define $F : M \rightarrow |K|$ by

$$F(x) = (f_1(x), \dots, f_N(x), \dots).$$

Let K' be the barycentric subdivision of K . Recall that $U_{i_0 \dots i_m} := U_{i_0} \cap \dots \cap U_{i_m} = \{\varphi_{i_0 \dots i_m} > 0\}$, where $\varphi_{i_0 \dots i_m} := \min_{j=0}^m \varphi_{i_j}$, and that $U_{i_0} \cap \dots \cap U_{i_m}$ is SLC to the unique maximum point, denoted by $p_{i_0 \dots i_m}$, of $\psi_{i_0 \dots i_m}$ via the gradient curves of $\psi_{i_0 \dots i_m}$. We now define $G : |K'| \rightarrow M$ as follows. For any $\sigma = [e_{i_0}, \dots, e_{i_k}] \in K$, let $b(\sigma)$ be the barycenter of σ . We put $G(b(\sigma)) := p_{i_0 \dots i_k}$. Assume that G is defined on the $(m-1)$ -skelton $|(K')^{m-1}|$ of K' in such a way that if all the vertices of an $(m-1)$ -simplex τ of K' is mapped via G to $U_{k_0 \dots k_\ell}$, $G(\tau)$ is also contained in $U_{k_0 \dots k_\ell}$. Now take any m -simplex $s = [b_0 \dots b_m]$ of K' . Let $U_{j_0}, \dots, U_{j_\ell}$ be the set containing all of $G(b_0), \dots, G(b_m)$. By the inductive assumption $G(\partial s)$ is also contained in $U_{j_0 \dots j_\ell}$. Lemma 4.2 enables us to extend $G : \partial s \rightarrow U_{j_0 \dots j_\ell}$ to a Lipschitz map $G : s \rightarrow U_{j_0 \dots j_\ell}$ by deforming $G(\partial s)$ to $p_{j_0 \dots j_\ell}$. Repeating this procedure, we have a Lipschitz map $G : |K'| \rightarrow M$.

Assertion 4.3. $F \circ G$ is Lipschitz homotopic to the identity 1_K .

Proof. For any $x = (x_1, \dots, x_N, \dots) \in |K'|$, let s and σ be the open simplexes of K' and K respectively containing x . Let $\sigma = [e_{i_0} \dots e_{i_\ell}]$. Take $0 \leq j \leq \ell$ with $G(x) \in U_{i_j}$. It follows that $f_{i_j}(G(x)) > 0$. Note $x_{i_j} > 0$. Set $h_i(x) = \min\{x_i, f_i(G(x))\}$, $1 \leq i < \infty$. Note $\sum_{i=1}^\infty h_i(x) > 0$ and define $H : |K'| \rightarrow |K|$ by

$$H(x) = \left(\frac{h_i(x)}{\sum_{i=1}^\infty h_i(x)} \right)$$

Since x and $H(x)$ as well as $H(x)$ and $F \circ G(x)$ are in the same simplex, $1|_{|K|}$ is Lipschitz homotopic to H , and H is Lipschitz homotopic to $F \circ G$. \square

Assertion 4.4. G is homotopy equivalent with homotopy inverse F .

Proof. From Assertion 4.3, G induces injective homomorphisms $\pi_*(|K|) \rightarrow \pi_*(M)$ in all dimensions. Note that M has the homotopy type of a CW -complex L of finite dimension since M is locally contractible and finite dimensional. For each $m \geq 1$ and each map $\alpha : S^m \rightarrow M$, set $\beta := G \circ F \circ \alpha : S^m \rightarrow M$. From construction, for each $x \in S^m$, if $F(\alpha(x))$ is contained in an open simplex $\sigma = (e_{i_0}, \dots, e_{i_k})$, then both $\alpha(x)$ and $\beta(x)$ are contained in some U_{i_j} with $0 \leq j \leq k$. Take a sufficiently fine triangulation Σ of S^m . If $\alpha(v)$ and $\beta(v)$ is in U_i for a vertex $v \in \Sigma^0$, we can join $\alpha(v)$ to $\beta(v)$ by a homotopy in U_i . Since Σ is sufficiently fine, this homotopy can be extended inductively to a homotopy between α and β on each skeleton Σ^ℓ with $0 \leq \ell \leq m$. Namely α is homotopic to β , and thus G induces isomorphisms $\pi_m(|K|) \rightarrow \pi_m(M)$

for all m . Therefore Whitehead's theorem implies that g is homotopy equivalent. \square

\square

Remark 4.5. In the situation of Theorem 1.1, M actually has the same *Lipschitz* homotopy type as the nerve of any good covering of it. The proof will appear in a forthcoming paper.

5. STABILITY OF GOOD COVERINGS

Let us recall that $\mathcal{A}(n, D, v_0)$ denote the set of all isometry classes of n -dimensional compact Alexandrov spaces M with curvature ≥ -1 , $\text{diam}(M) \leq D$, $\text{vol}(M) \geq v_0 > 0$. In this section, we prove Theorem 1.2.

The construction of locally defined strictly concave functions is stable in the non-collapsing convergence as follows.

Lemma 5.1 ([10], [13]). *Let $M \in \mathcal{A}(n, D, v)$ and $M_j \in \mathcal{A}(n, D, v)$ a sequence converging to M as $j \rightarrow \infty$. For any $p \in M$, there exist $r > 0$ a strictly concave function φ defined on $U(p, r)$, and a strictly concave function φ_j defined on $U(p_j, r)$, for large j and for some $p_j \in M_j$ converging to p under the convergence $M_j \rightarrow M$ such that φ_j converges to φ . Further, p is the unique maximum of φ and p_j is the unique maximum of φ_j .*

Proof of Theorem 1.2. Let us fix $M \in \mathcal{A}(n, D, v)$. We construct a finite good cover $\mathcal{U}_M = \{U_i\}_{i=1}^N$ of M as in the proof of Theorem 1.1(2). Let us recall that each U_i has the form $U_i = \{x \in U(p_i, r_i) \mid \varphi_i > 0\}$ for some $p_i \in M$, $r_i > 0$ and a strictly concave proper function φ_i defined on $U(p_i, r_i)$.

For $T \subset \{1, \dots, N\}$, we set $U_T = \bigcap_{i \in T} U_i$ and if U_T is nonempty, then we set $m_T := \max_{U_T} \varphi_T$, where $\varphi_T = \min_{i \in T} \varphi_i$ which is strictly concave on U_T . Further, we set $m := \min\{m_T \mid U_T \neq \emptyset\}$. We may assume that setting $V_i = \{x \in U(p_i, r_i) \mid \varphi_i(x) > \delta_0 m\}$ for some $\delta_0 > 0$, the family $\{V_i\}_{i=1}^N$ still covers M . Let $\mathcal{U}_M = \{V_i\}_{i=1}^N$. By the construction and Theorem 1.4, \mathcal{U}_M is also a good covering of M .

By Lemma 5.1, there is $\epsilon_0 > 0$ depending on n, D, v_0 such that for any $M' \in \mathcal{A}(n, D, v_0)$ with $d_{\text{GH}}(M, M') < \epsilon_0$, there exist $p'_i \in M'$ and strictly concave proper functions φ'_i on $U(p'_i, r_i)$ which is regular on $U(p'_i, r_i) \setminus \{p'_i\}$. Here, each p'_i is close to p_i via a Gromov-Hausdorff approximation between M and M' and the radius of $U(p'_i, r_i)$ is the same as $U(p_i, r_i)$. Set $\psi : M \rightarrow M'$ and $\psi' : M' \rightarrow M$ ϵ -Gromov-Hausdorff approximations, where $\epsilon = d_{\text{GH}}(M, M')$. Further, by Lemma 5.1, we have

$$|\varphi_i(x) - \varphi'_i(\psi(x))| < \epsilon_1 \text{ and } |\varphi_i(\psi'(y)) - \varphi'_i(y)| < \epsilon_1$$

for all $x \in U(p_i, r_i)$ and $y \in U(p'_i, r_i)$, where $\epsilon_1 > 0$ satisfies $\lim_{\epsilon \rightarrow 0} \epsilon_1 = 0$ when r_i is fixed. When ϵ_0 is so small, setting $V'_i = \{x \in U(p'_i, r_i) \mid$

$\varphi'_i(x) > \delta_0 m/2\}$, the family $\{V'_i\}_{i=1}^N$ covers M' . Then, by Lemma 4.2, the covering $\{V'_i\}$ of M' is good. For any subset $T \subset \{1, \dots, N\}$, we set $V_T = \bigcap_{i \in T} V_i$ and $V'_T = \bigcap_{i \in T} V'_i$. By the construction, for every $x \in V_i$, we have $\psi(x) \in V'_i$. Hence, if V_T is nonempty, then V'_T is nonempty. Conversely, let us take $y \in V'_T$. Then, $\varphi_i(\psi'(y)) > \delta_0 m/3$ for all $i \in T$. Hence, $\psi'(y) \in U_T$. By the choice of m , along the gradient curve of φ_T starting from $\psi'(y)$, we can find a point $x \in U_T$ with $\varphi_T(x) > \delta_0 m$. Hence, V_T is nonempty. Therefore, the nerves of $\{V_i\}_{i=1}^N$ and $\{V'_i\}_{i=1}^N$ are isomorphic.

This argument implies that a value ϵ_M is positive for each $M \in \mathcal{A}(n, D, v_0)$, where

$$\epsilon_M := \sup \left\{ \epsilon > 0 \mid \begin{array}{l} \text{There is a finite simplicial complex } K \text{ such} \\ \text{that every } M' \in U_{\text{GH}}(M, \epsilon) \text{ admits a good} \\ \text{covering whose nerve is isomorphic to } K \end{array} \right\}$$

Here, $U_{\text{GH}}(M, \epsilon)$ denotes the ϵ -neighborhood of M in $\mathcal{A}(n, D, v)$. Since $\mathcal{A}(n, D, v_0)$ is compact, $\inf_{M \in \mathcal{A}(n, D, v_0)} \epsilon_M > 0$. Hence, there are $\epsilon_0 > 0$, finitely many $M_1, \dots, M_N \in \mathcal{A}(n, D, v_0)$ and finite simplicial complexes K_1, \dots, K_N such that $\bigcup_{i=1}^N U_{\text{GH}}(M_i, \epsilon_0) = \mathcal{A}(n, D, v_0)$ and that any $M \in U_{\text{GH}}(M_i, \epsilon_0)$ has a good covering whose nerve is isomorphic to K_i for each i . This completes the proof. \square

6. APPENDIX: A VERSION OF FIBRATION THEOREM

Fibration Theorem 1.5 is important to determine the topological structure of Alexandrov spaces via regular functions. Actually, Perelman proved it for admissible functions on spaces with or without boundary in [16] and [17], and for general semiconcave functions on spaces without boundary in [18]. In this section, we prove Theorem 1.5 and its generalization (Theorem 6.6 stated later), which are versions of fibration theorems for semiconcave functions on spaces with non-empty boundary.

6.1. MCS-spaces. To prove Theorems 1.5 and 6.6 stated below, we recall the notion of *MCS-spaces* introduced by Perelman ([16], [17]). Those spaces are defined inductively: 0-dimensional MCS-spaces are defined to be discrete sets; a separable metrizable space X is called an n -dimensional MCS-space if for any $x \in X$, there exist an open neighborhood U and a compact $(n-1)$ -dimensional MCS-space Y such that (U, x) is homeomorphic to $(K(Y), o)$ as pointed spaces, where o denotes the apex of the cone. We call U a conical neighborhood of x and Y a generator of U . Any MCS-space has a canonical stratification into topological manifolds, as follows. Let X be an n -dimensional MCS-space. Then, X has a canonical stratification into topological manifolds $\{X^{(\ell)}\}_{\ell=0}^n$ by the following way: $p \in X$ is in the (canonical) ℓ -stratum $X^{(\ell)}$ if p has a conical neighborhood homeomorphic to $\mathbb{R}^\ell \times K$, where ℓ

is taken to be maximal and K is a cone over some compact MCS-space. Then, $X^{(\ell)}$ is a topological ℓ -manifold. We call $X^{(n)}$ the *top stratum* of X . Note that $X^{(n)}$ is always non-empty and dense in X .

Siebenmann proved the following important theorem in the category of topological spaces:

Theorem 6.1 ([24]). *Let $f : X \rightarrow Y$ be a proper continuous surjection between topological spaces. Suppose that*

- (A) *for each $y \in Y$, the fiber $f^{-1}(y)$ is an MCS-space;*
- (B) *f is a topological submersion, that is, for any $x \in X$, there exist an open neighborhood V of x and a homeomorphism $\varphi : (f^{-1}(f(x)) \cap V) \times f(V) \rightarrow V$ respecting f . Here, the letter means that $f \circ \varphi$ is the projection of the second coordinate of the product.*

Then, f is a fiber bundle, that is, for any $y \in Y$, there exist an open neighborhood U of y and a homeomorphism $\theta : f^{-1}(y) \times U \rightarrow f^{-1}(U)$ respecting f .

We call such a map φ in the condition (B) a product chart of f at x , and a V a product neighborhood. The theorem actually holds if we replace an MCS-space with a space having some property about deformations of homeomorphisms. For instance, as in [24], CS sets and WCS sets have such a property. MCS-spaces give a middle class of between CS and WCS sets.

6.2. The original fibration theorem by Perelman. Let Σ denotes an Alexandrov space of curvature ≥ 1 . A semiconcave function $g : \Sigma \rightarrow \mathbb{R}$ is said to be spherically concave if it is g -concave, or equivalently, the cone extension $K(g) : K(\Sigma) \rightarrow \mathbb{R}$ is concave, where $K(g)(a\xi) = ag(\xi)$ for $a \geq 0$ and $\xi \in \Sigma$. The inner product $\langle g, h \rangle$ of two spherically concave functions g, h on Σ is defined by

$$\langle g, h \rangle := \sup_{\xi \in \Sigma} (g(\xi)h(\xi) + \langle g'_\xi, h'_\xi \rangle).$$

Here, 0-dimensional Alexandrov spaces are the spaces of single point or consisting of two points of distance π , and hence, the inner product is actually defined inductively. Note that, the derivation of every semiconcave function is spherically concave on the space of directions at each point.

Let U be an open set of an Alexandrov space M having no boundary points. A map $f = (f_1, \dots, f_k)$ consisting of semiconcave functions f_i defined on U is said to be regular at $x \in U$ if

- there is $\xi \in \Sigma_x$ such that $(f_i)'_x(\xi) > 0$ for each i and
- $\langle (f_i)'_y, (f_j)'_y \rangle < 0$ for y near x and for every $i \neq j$.

The set of all regular points in U is open. The original fibration theorem for semiconcave functions is stated as follows.

Theorem 6.2 ([18]). *Let $f = (f_1, \dots, f_k)$ be a map consisting of semi-concave functions on U as above. Suppose that f is regular on U . Then, the following holds.*

- $f(U)$ is open in \mathbb{R}^k ;
- each fiber of f is an $(n - k)$ -dimensional MCS-space;
- $f : U \rightarrow f(U)$ is a topological submersion;
- further, if $f : U \rightarrow f(U)$ is proper, that is, the preimage of every compact set in $f(U)$ is compact, then it is a fiber bundle.

We note that the function $f(x) = -|x|^2/2$ is (-1) -concave on a disk $D = \{x \in \mathbb{R}^2 \mid |x - 1| \leq 1\}$ with the flat metric, but it is not admissible and its extension to the double is not semiconcave. Indeed, the derivation of f at $x_0 = 2$ a boundary point is of the form

$$f'_{x_0}(\xi) = 2 \cos \angle(0'_{x_0}, \xi)$$

for $\xi \in \Sigma_{x_0}(D)$, which is not contained in the class of DER functions (see [17]). Hence, f is not admissible. Further, every point except 0 is a regular point of f , but the fibration theorem does not hold for f , because two regular fibers $f^{-1}(f(x_0)) = \{x_0\}$ and $f^{-1}(f(1))$ are not homeomorphic.

Perelman's Stability Theorem ([16], [11]) is very important in the geometry of Alexandrov spaces. The proof of it is based on the fibration theorem (for admissible functions). The fibration Theorems states that each regular fiber is a general MCS-space. Further, we can prove that each fiber of dimension one is actually a manifold, by using Stability Theorem as follows.

Lemma 6.3. *Let U be an open subset in an n -dimensional Alexandrov space having no boundary point and $f = (f_i) : U \rightarrow \mathbb{R}^{n-1}$ a map consisting of semiconcave functions f_i . Suppose that f is regular on U . Then, the fiber of f at each point in $f(U)$ is a one-manifold without boundary.*

Proof. We may assume that $n \geq 2$. The original fibration theorem states that the fiber $F = f^{-1}(v)$ at $v \in f(U)$ is a one-dimensional MCS-space, which is a locally finite graph in general. Let $x \in F$ be a vertex of the graph. Since f is a topological submersion, x has a conical neighborhood V in U such that (V, x) is homeomorphic to $(K(\Lambda) \times \mathbb{R}^{n-1}, o)$, where Λ is a link at x in the graph F . Then, we have

$$H^n(U, U \setminus x) \cong \bar{H}^0(\Lambda).$$

Here, the cohomologies are considered having \mathbb{Z}_2 -coefficients. On the other hands, by Stability Theorem and by Grove and Petersen ([7]), we have

$$H^n(U, U \setminus x) \cong H^{n-1}(\Sigma_x) \cong \mathbb{Z}_2.$$

Therefore, the set Λ consists of only two points. Consequently, F does not have a branching point, that is, F is a one-manifold without boundary. This completes the proof. \square

6.3. Double. Let M denote an Alexandrov space with nonempty boundary. Its double $D(M)$ is defined as the quotient space of the disjoint union $M \amalg M$ by identifying boundary points of two M 's. Perelman proved that $D(M)$ is also an Alexandrov space with the canonical metric ([16]). Then, M is regarded as an isometrically embedded subset of $D(M)$. For any $x \in D(M)$, we set $r(x)$ the point corresponding to x in the other copy of M in $D(M)$. This r defines a canonical isometric involution (reflection via the boundary) on $D(M)$, and the fixed point set is equal to ∂M . For any subset S of M , we set $D(S) = S \cup r(S) \subset D(M)$. For a map $g : S \rightarrow Y$ to a space Y , its canonical extension to the double $D(S)$ is defined by $\tilde{g}(x) = \tilde{g}(r(x))$ for $x \in S$. For $x \in \partial M$, $\tilde{B}(x, \delta)$ denotes the closed ball in $D(M)$ centered at x of radius δ .

6.4. Equivariant incomplementable lemma. Let U be an open subset of an n -dimensional Alexandrov space M such that $U \cap \partial M \neq \emptyset$. Let $f : U \rightarrow \mathbb{R}^k$ be a map such that the double extension \tilde{f}_i is semiconcave for each component f_i , where $k < n$. Let $p \in U \cap \partial M$ be such that $\tilde{f} = (\tilde{f}_i)$ is regular at p . We say that \tilde{f} is r -complementable at p if there exist an open neighborhood V of p in U and a Lipschitz function f_{k+1} defined on V such that the extension \tilde{f}_{k+1} of f_{k+1} is semiconcave on $D(V)$ and $(\tilde{f}, \tilde{f}_{k+1})$ is regular at p . Otherwise, we say that f is r -incomplementable at p .

Lemma 6.4. *Let f, U and M be as above. Suppose that \tilde{f} is regular at p and f is r -incomplementable at p for $p \in U \cap \partial M$. Then, there exist an open neighborhood V at p in U , a Lipschitz function g defined on V and a Lipschitz function $H : f(V) \rightarrow \mathbb{R}$ such that \tilde{g} is semiconcave on $D(V)$;*

- (a) *a map $\begin{pmatrix} \text{id}_{\mathbb{R}^k} & 0 \\ -H & \text{id}_{\mathbb{R}} \end{pmatrix} : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ is bi-Lipschitz on $(f, h)(V)$;*
- (b) *setting an r -equivariant function $\tilde{h} := \tilde{g} - H \circ \tilde{f}$, we have $\tilde{h}(p) = 0$ and $\tilde{h} \leq 0$ on $D(V)$;*
- (c) *(\tilde{f}, \tilde{g}) is regular on $\tilde{K}_\rho \setminus \tilde{h}^{-1}(0)$;*
- (d) *$\tilde{f}^{-1}(v) \cap \tilde{h}^{-1}(0) \cap \tilde{K}_\rho$ is a single-point set for each $v \in \tilde{f}(\tilde{K}_\rho)$.*

Here, \tilde{K}_ρ is a compact set defined as

$$\tilde{K}_\rho = \{x \in D(V) \mid \|\tilde{f}(x) - f(p)\| \leq \rho, \tilde{h}(x) \geq -2\rho\}$$

for small $\rho > 0$. Here, the norm on \mathbb{R}^k is the maximum norm.

All the statements follows from the proof of the original corresponding statements in [16] and [17] with some modification.

Proof. We are going to show that all objects appeared in the original corresponding statements in [17], [16] can be constructed in an equivariant way, in our case.

Since \tilde{f} is regular at p , there is a direction $\xi \in \Sigma_p D(M)$ such that $(\tilde{f}_i)'_p(\xi) > \epsilon$ for all i , where ϵ is a positive number. Note that such a ξ can be assumed to be in $\Sigma_p(\partial M)$. Around p , we may assume that all \tilde{f}_i are λ -concave for some $\lambda > 0$. Take a point q in the direction ξ , we have

$$\tilde{f}_i(q) > \tilde{f}_i(p) + \epsilon|pq| + \lambda|pq|^2$$

for all $i = 1, \dots, k$. We may assume that $q \in \partial M$. For some $\delta > 0$ smaller than a constant depending on $\epsilon, \lambda, |pq|$, we have

$$\tilde{f}_i(y) > \tilde{f}_i(x) + (\epsilon/4)|xy| + (\lambda/2)|xy|^2$$

for all $x \in \tilde{B}(p, \delta)$ and $y \in \tilde{B}(q, \epsilon|pq|/4)$. For a fine discrete set $\{q_\alpha\}_{\alpha=1}^N \subset \tilde{B}(q, \epsilon|pq|/4) \cap \partial \tilde{B}(p, |pq|)$, the function

$$\tilde{g} = \frac{1}{N} \sum_{\alpha=1}^N \chi(|q_\alpha \cdot |)$$

is strictly concave on $\tilde{B}(p, \delta)$, for some concave increasing real-to-real function χ which is strictly concave near $|pq|$. Now, we should note that $\{q_\alpha\}$ can be taken to be r -invariant. Hence, the function \tilde{g} is r -equivariant. Let us set $V := B(p, \delta)$ and g the restriction of \tilde{g} to V . We define a function $H : f(V) \rightarrow \mathbb{R}$ by $H(v) = \max\{g(x) \mid x \in V \cap f^{-1}(v)\} = \max\{\tilde{g}(x) \mid x \in D(V) \cap \tilde{f}^{-1}(v)\}$. Then, a function

$$\tilde{h} = \tilde{g} - H \circ \tilde{f}$$

is r -equivariant. From the proof of the original statements, all properties (a), (b), (c) and (d) hold, for constructed V, g and H in our case. We leave the complete proof to readers. \square

We give several immediate consequences of Lemma 6.4.

By the property (c) in Lemma 6.4 and the compactness of \tilde{K}_ρ , the map $(\tilde{h}, \tilde{f}) : \tilde{K}_\rho \setminus \tilde{h}^{-1}(0) \rightarrow [-2\rho, 0] \times D^k(\rho)$ is a fiber bundle, by the original Fibration Theorem 6.2. Here, $D^k(\rho)$ is given as

$$D^k(\rho) = \tilde{f}(\tilde{K}_\rho) = \{v \in \mathbb{R}^k \mid \|v - f(p)\| \leq \rho\}.$$

Since the image $[-2\rho, 0] \times D^k(\rho)$ is contractible, there is a homeomorphism

$$(6.5) \quad \varphi : \tilde{K}_\rho \setminus \tilde{h}^{-1}(0) \rightarrow \tilde{\Pi}_\rho \times [-2\rho, 0] \times D^k(\rho)$$

respecting (\tilde{h}, \tilde{f}) . Here, $\tilde{\Pi}_\rho$ is a regular fiber of (\tilde{h}, \tilde{f}) in \tilde{K}_ρ given as

$$\tilde{\Pi}_\rho = \tilde{f}^{-1}(f(p)) \cap \tilde{h}^{-1}(-\rho) \cap \tilde{K}_\rho$$

which is an $(n - k)$ -dimensional compact MCS-space, by the original fibration theorem ([18]). The set $\tilde{\Pi}_\rho$ is r -invariant, by the definition.

We prepare some symbols which will be used later. Let us set $K_\rho = \tilde{K}_\rho \cap U$, $\Pi_\rho = \tilde{\Pi}_\rho \cap U$, and $h = g - H \circ f$.

Note that by (d), we have $\tilde{K}_\rho \cap \tilde{h}^{-1}(0) \subset \partial M$. Indeed, the set $\tilde{f}^{-1}(v) \cap \tilde{K}_\rho \cap \tilde{h}^{-1}(0)$ is r -invariant, by the definition, and is a single-point set by (d). Therefore, it is contained in ∂M . Remark that, $\tilde{K}_\rho \cap \tilde{h}^{-1}(0)$ does not coincide with $\tilde{K}_\rho \cap \partial M$, in general.

6.5. A fibration theorem in our case. From now on, we fix the following situation. Let f_1, \dots, f_k be locally Lipschitz functions defined on an open subset U of an n -dimensional Alexandrov space M with boundary. We consider the case $U \cap \partial M \neq \emptyset$ and the canonical extension \tilde{f}_i on $D(U)$ is semiconcave for every i .

Lemma 6.5. *Let $f = (f_i)$, U and M be as above. Suppose that $\tilde{f} = (\tilde{f}_i)$ is regular at some $p \in D(U) \cap \partial M$. Then, we have $k \leq n - 1$. Further, if \tilde{f} is regular on U , then $f : U \rightarrow \mathbb{R}^k$ is an open map.*

Proof. Since \tilde{f} is regular at p , by [18], we have $k \leq n$. We suppose $k = n$. Let $p \in U \cap \partial M$. The original fibration theorem states that \tilde{f} is homeomorphic near p in $D(U)$. That is, there is a neighborhood V of p in U such that $\tilde{f} : D(V) \rightarrow \mathbb{R}^n$ is an embedding. The restriction $f : V \rightarrow \mathbb{R}^n$ is also an embedding. However, because \tilde{f} is r -equivariant, the images $f(V)$ and $\tilde{f}(D(V))$ coincide. It is a contradiction. Hence, we have $k \leq n - 1$.

Suppose that \tilde{f} is regular on $D(U)$. By Theorem 6.2, the map $\tilde{f} : D(U) \rightarrow \mathbb{R}^k$ is an open map. For any open set O in U , its double $D(O)$ is open in $D(U)$. Therefore, the map $f : U \rightarrow f(U)$ is also an open map. Hence, the letter conclusion is proved. \square

Theorem 6.6. *Let $f = (f_i)$, U and M be as above. Suppose that $\tilde{f} = (\tilde{f}_i)$ is regular on $D(U)$. Then, we have*

- (A) *For every $v \in f(U)$, its fiber $f^{-1}(v)$ is an $(n - k)$ -dimensional MCS-space.*
- (B) *$f : U \rightarrow f(U)$ is a topological submersion.*
- (C) *If $f : U \rightarrow f(U)$ is proper, then it is a fiber bundle.*

Proof. The properties (A), (B) and (C) for k , where k is the dimension of the target of f , are denoted by $(A)_k$, $(B)_k$ and $(C)_k$. We prove the properties by the backward induction on k as the proof of the original fibration theorem. By Theorem 6.1, if $(A)_k$ and $(B)_k$ hold, then $(C)_k$ holds.

Let us prove $(A)_{n-1}$ and $(B)_{n-1}$. To prove them, we find a product neighborhood at a point $p \in U \cap \partial M$ with respect to f . Note that every point in $U \setminus \partial M$ already has a product neighborhood, by the original fibration theorem ([18]). By Lemma 6.5, f is r -incomplementable at p . Then, there exist an open neighborhood V of p in U , a Lipschitz

function g defined on V and a Lipschitz function $H : f(V) \rightarrow \mathbb{R}$ satisfying the conclusion of Lemma 6.4. By using them, we have a product chart at p in \tilde{K}_ρ with respect to (\tilde{f}, \tilde{h}) as the following way. First, by properties (a) and (c) in Lemma 6.4 and by the original fibration theorem, we obtain a homeomorphism

$$\varphi : \tilde{K}_\rho \setminus \tilde{h}^{-1}(0) \rightarrow \tilde{\Pi}_\rho \times [-2\rho, 0) \times D^{n-1}(\rho)$$

respecting (\tilde{h}, \tilde{f}) as in (6.5). By the property (d), we have a canonical extension ψ of φ

$$\psi : \tilde{K}_\rho \rightarrow \bar{K}(\tilde{\Pi}_\rho) \times D^{n-1}(\rho)$$

which is a homeomorphism respecting \tilde{f} . Let us set $\tilde{F}_v = \tilde{f}^{-1}(v) \cap \tilde{K}_\rho$ and denote by $p(v)$ the unique point contained in $\tilde{F}_v \cap \tilde{h}^{-1}(0)$ for $v \in D^{n-1}(\rho)$. In particular, $p(f(p)) = p$. Then, we have $(\tilde{F}_v, p(v))$ is homeomorphic to $(\bar{K}(\tilde{\Pi}_\rho), o)$. Since the relative interior of the fiber \tilde{F}_v is a one-manifold without boundary, $\tilde{\Pi}_\rho$ must be a two-points set. Because $\tilde{\Pi}_\rho$ is r -invariant, the set Π_ρ consists of only one point. We observe that $\tilde{F}_v \cap \tilde{h}^{-1}(0) = \tilde{F}_v \cap \partial M$. Indeed, by the remark after Lemma 6.4, we know that $\tilde{F}_v \cap \tilde{h}^{-1}(0) \subset \partial M$. We suppose that \tilde{F}_v meets ∂M at least two points. Then, since \tilde{F}_v is r -invariant, \tilde{F}_v contains a circle, which contradicts to that \tilde{F}_v is an interval. Therefore, we have $\tilde{F}_v \cap \tilde{h}^{-1}(0) = \tilde{F}_v \cap \partial M$. Let us set $F_v = f^{-1}(v) \cap K_\rho$. We conclude that $(\tilde{F}_v, F_v, p(v))$ is homeomorphic to $([-1, 1], [0, 1], 0)$ as triples of spaces. In particular, (A) _{$n-1$} holds. Further, we know that the restriction of φ to $K_\rho \setminus \tilde{h}^{-1}(0) = K_\rho \setminus \partial M$ has the image $\Pi_\rho \times [-2\rho, 0) \times D^{n-1}(\rho)$. Therefore, the restriction of ψ to K_ρ is a homeomorphism with the target $\bar{K}(\Pi_\rho) \times D^{n-1}(\rho)$ respecting f . It gives a product chart at p of f . This completes the proof of (B) _{$n-1$} .

Next, we are going to prove (A) _{k} and (B) _{k} for $k < n - 1$, assuming (A) _{$k+1$} , (B) _{$k+1$} and (C) _{$k+1$} . Let $F = f^{-1}(v)$ for $v \in f(U)$. We already know that every point in $F \setminus \partial M$ has a conical neighborhood as an MCS-space, by the original fibration theorem ([18]). We may assume that $F \cap \partial M$ is not the empty-set, and take a point p in the intersection. To prove (A) _{k} and (B) _{k} , we find a conical neighborhood at p in F as an MCS-space and a product neighborhood at p with respect to f . If f is r -complementable at p , then there is a function f_{k+1} defined near p such that \tilde{f}_{k+1} is semiconcave and $(\tilde{f}, \tilde{f}_{k+1})$ is regular at p in the double of a neighborhood of p . Then, by (B) _{$k+1$} , we have a product chart

$$\theta : V \rightarrow (f^{-1}(f(p)) \cap f_{k+1}^{-1}(f_{k+1}(p)) \cap V) \times (f, f_{k+1})(V)$$

at p of (f, f_{k+1}) . Taking V to be small, we may assume that the image of (f, f_{k+1}) is the form $(f, f_{k+1})(V) = f(V) \times (a, b)$, where $f_{k+1}(V) = (a, b)$. Let us set $F := f^{-1}(f(p)) \cap f_{k+1}^{-1}(f_{k+1}(p)) \cap V$, which is an $(n - k - 1)$ -dimensional MCS-space. We obtain a product chart

$$\eta : V \rightarrow (f^{-1}(f(p)) \cap V) \times f(V)$$

of f . Then, $f^{-1}(f(p)) \cap V$ is homoeomorphic to $F \times (a, b)$, which is an $(n - k)$ -dimensional MCS-space. Therefore, in this case, $(A)_k$ and $(B)_k$ are proved.

We next assume that f is r -incomplementable at p . Then, there exist a neighborhood V of p , a function g defined on V and a function $H : f(V) \rightarrow \mathbb{R}$ satisfying the conclusion of Lemma 6.4. Since (\tilde{f}, \tilde{g}) is regular on $\tilde{K}_\rho \setminus \tilde{h}^{-1}(0)$, by $(C)_{k+1}$, we have a homeomorphism

$$\varphi : K_\rho \setminus \tilde{h}^{-1}(0) \rightarrow \Pi_\rho \times [-2\rho, 0] \times D^k(\rho)$$

respecting (h, f) . The space Π_ρ is an $(n - k - 1)$ -dimensional MCS-space, by $(A)_{k+1}$. By (d) in Lemma 6.4, φ has a canonical extension

$$\psi : K_\rho \rightarrow \bar{K}(\Pi_\rho) \times D^k(\rho)$$

which is a homeomorphism respecting f . Then, it gives a product chart at p with respect to f . This implies $(B)_k$. Further, by the construction of ψ , we have that $(f^{-1}(f(p)) \cap K_\rho, p)$ is homeomorphic to $(\bar{K}(\Pi_\rho), o)$. Therefore, $f^{-1}(f(p))$ is an $(n - k)$ -dimensional MCS-space. Hence, $(A)_k$ is proved. This completes the proof of Theorem 6.6. \square

Remark 6.7. As Lemma 6.3, we can prove that each fiber in Fibration Theorems 6.2 and 6.6 belongs to some restricted class of MCS-spaces, if k is general. For instance, it is represented as a non-branching MCS-space introduced in [9]. The proof will appear in a forthcoming paper.

REFERENCES

- [1] R. Bott and L. W. Tu. Differential Forms in Algebraic Topology. Graduate Studies in Mathematics, 82. Springer.
- [2] D. Burago, Yu. Burago, and S. Ivanov. A course in metric geometry. Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. xiv+415 pp.
- [3] Yu. Burago, M. Gromov, and G. Perel'man. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk* 47 (1992), no. 2(284), 3–51, 222, translation in *Russian Math. Surveys* 47 (1992), no. 2, 1–58.
- [4] J. Cheeger, Finiteness theorems for Riemannian manifolds, *Amer. J. Math.*, 92(1970), 61–74.
- [5] M. Gromov. Curvature, diameter and Betti numbers, *Comment. Math. Helv.* 56(1981), 179–195.
- [6] K. Grove and P. Petersen. Bounding homotopy types by geometry, *Ann. of Math.* 128(1988), 195–206.
- [7] K. Grove and P. Petersen. A radius sphere theorem, *Invent. Math.* 112 (1993), no. 3, 577–583.
- [8] K. Grove, P. Petersen and J. Y. Wu. Geometric finiteness theorems via controlled topology, *Invent. Math.* 99(1990), 205–213.
- [9] J. Harvey and C. Searle. Orientation and symmetries of Alexandrov spaces with applications in positive curvature, arXiv:1209.1366v3.
- [10] V. Kapovitch. Regularity of limits of noncollapsing sequences of manifolds. *Geom. Funct. Anal.* 12(1), 121–137 (2000).

- [11] V. Kapovitch. Perelman's stability theorem, Surveys of Differential Geometry, Metric and Comparison Geometry, vol. XI, International press, (2007), 103–136.
- [12] K. W. Kwun. Uniqueness of the open neighborhood. Proc. Amer. Math. Soc. 15 (1964), 476–479.
- [13] A. Mitsuishi and T. Yamaguchi Stability of strongly Lipschitz contractible balls in Alexandrov spaces, Math. Z. 277(2014),995-1009.
- [14] A. Mitsuishi and T. Yamaguchi Locally strongly Lipschitz contractibility of Alexandrov spaces, Pacific J. 270 (2014),393421.
- [15] G. de Rham. Complexes à automorphismes et homéomorphie différentiable, Ann. Inst. Fourier (Grenoble) 2 (1950), 5167.
- [16] G. Perelman. A. D. Alexandrov's spaces with curvatures bounded from below. II, Preprint.
- [17] G. Perelman. Elements of Morse theory on Alexandrov spaces, St. Petersburg Math. J. 5 (1994) 207–214.
- [18] G. Perelman. DC Structure on Alexandrov Space. Preprint.
- [19] G. Perelman and A. Petrunin. Quasigeodesics and gradient curves in Alexandrov spaces. Preprint.
- [20] P. Petersen. A finiteness theorem for metric spaces, J. Diff. Geometry, 31(1990), 387-395.
- [21] A. Petrunin. Quasigeodesics in multidimensional Alexandrov spaces. Thesis (Ph.D.)-University of Illinois at Urbana-Champaign. 1995. 92 pp.
- [22] A. Petrunin. Semiconcave functions in Alexandrov's geometry. Surveys in Comparison Geometry, 2007.
- [23] G. de Rham. Complexes à automorphismes et homéomorphie différentiable, Ann. Inst. Fourier (Grenoble)2 (1950), 51-67.
- [24] L. C. Siebenmann. Deformation of homeomorphisms on stratified sets. I, II. Comment. Math. Helv. 47 (1972), 123–136; ibid. 47 (1972), 137–163.
- [25] A. Weinstein. On the homotopy types of positively pinched manifolds, Arch. Math. 18(1967), 523-524.
- [26] T. Yamaguchi. Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds, Proc. Amer. Math. Soc., 102(1988), 660-666.
- [27] T. Yamaguchi. Collapsing and essential coverings, arXiv:1205.0441.

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